Problem Set 8 due November 11, at 8 PM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Consider the matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

(1) Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ by computing them explicitly. *(10 points)*

(2) Use the previous part to prove that $\mathrm{Tr}(D) = \mathrm{Tr}(VDV^{-1})$ for any square matrix D and any invertible matrix V. *(5 points)*

Problem 2: Consider any square matrices A, B, V with the latter being invertible, such that:

$$B = VAV^{-1}$$

(1) Prove that $B - \lambda I = V(A - \lambda I)V^{-1}$. *(5 points)*

(2) Use the previous part to prove that:

$$\det(B - \lambda I) = \det(A - \lambda I)$$

(5 points)

(3) Use the previous part to prove that the conjugate matrices A and B have the same eigenvalues. *(5 points)*

(4) Use the previous part to give another proof of problem 1.2. *(5 points)*

Problem 3: Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Prove that the any non-zero eigenvalue of the square matrix AB is also an eigenvalue of the matrix BA. *(10 points)*

Problem 4: Consider the matrix:

$$A = \begin{bmatrix} -4 & -3 & -2 \\ 3 & 3 & 1 \\ 15 & 8 & 7 \end{bmatrix}$$

(1) Find an eigenvalue λ of A, and compute an eigenvector $\boldsymbol{v}$. *(10 points)*

(2) Compute a vector $\boldsymbol{w}$ such that $(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$ and a vector $\boldsymbol{z}$ such that $(A - \lambda I)\boldsymbol{z} = \boldsymbol{w}$. *(10 points)*

(3) Consider the matrix $V = \begin{bmatrix}\boldsymbol{v}|\boldsymbol{w}|\boldsymbol{z}\end{bmatrix}$ and compute:

$$V^{-1}AV$$

Congratulations: you just computed the Jordan normal form of A. *(10 points)*

Problem 5: Consider the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(1) Compute $A^2, A^3, A^4, A^5, ...$ and write a formula for the matrix exponential e^{At}. *(10 points)*

(2) Use formula (205) in the notes to find the solution of the system $\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)$. *(10 points)*

(3) Find the complete solution to the differential equation $f'''(t) = 0$ by setting it up as the system of differential equations in part (2). *(5 points)*